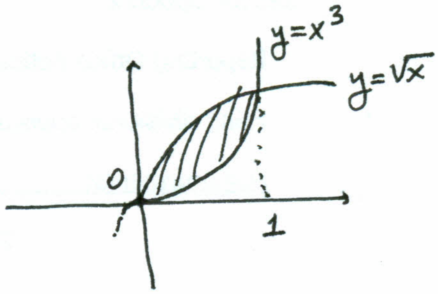
Exercises & Review Problems

1) Evaluate a) $\iint\limits_R (xy - y^3)\,dA$, R is the region bounded by $y=\sqrt{x}$ and $y=x^3$.

<u>Solution</u>

$$= \int_0^1 \int_{x^3}^{\sqrt{x}} (xy - y^3)\,dy\,dx = \cdots$$



b) $\iint\limits_R (6x^2 - 40y)\,dA$, R is the triangle with vertices $(0,3)$, $(1,1)$ and $(5,3)$.

<u>Solution</u>

$$\int_0^1 \int_{-2x+3}^{3} (6x^2 - 40y)\,dy\,dx + \int_1^5 \int_{\frac{x+1}{2}}^{3} (6x^2 - 40y)\,dy\,dx = \cdots$$

or

$$\int_1^3 \int_{\frac{-y+3}{2}}^{2y-1} (6x^2 - 40y)\,dx\,dy = \cdots = -\frac{935}{3}$$

2) Evaluate the following integrals by first reversing the order of integration:

a) $\displaystyle\int_0^3 \int_{x^2}^{9} x^3 e^{y^2}\,dy\,dx$ b) $\displaystyle\int_0^8 \int_{\sqrt[3]{y}}^{2} \sqrt{x^4+1}\,dx\,dy$

<u>Solutions</u>

a) $$= \int_0^9 \int_0^{\sqrt{y}} x^3 e^{y^2}\,dx\,dy = \cdots = \frac{1}{12}\left(e^{729} - 1\right)$$

b) $$= \int_0^2 \int_0^{x^3} \sqrt{x^4+1}\,dy\,dx = \cdots = \frac{1}{6}\left(17^{3/2} - 1\right)$$

3) Evaluate $$\int_0^1 \int_0^{\sqrt{1-y^2}} \cos(x^2+y^2)\,dx\,dy$$

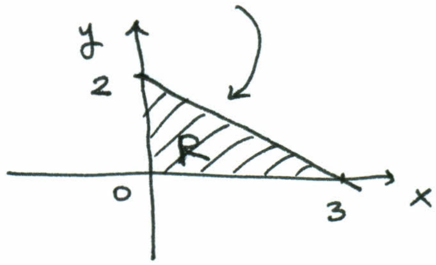
<u>Hint.</u> Use polar coordinates,

$$= \int_0^{\pi/2} \int_0^1 \cos(r^2)\, r\,dr\,d\theta = \cdots = \frac{\pi}{4}\sin 1$$

4) Evaluate $\iiint_S x\,dV$, where S is the region under the plane $2x+3y+z=6$ that lies in the first octant.

<u>Solution</u> The first octant means all three coordinates are positive.

<u>xy-projection</u> $(z=0) \Rightarrow 2x+3y=6 \Rightarrow y=-\frac{2}{3}x+2$



$$\therefore \iint_R \left[\int_0^{6-2x-3y} x\,dz\right] dA$$

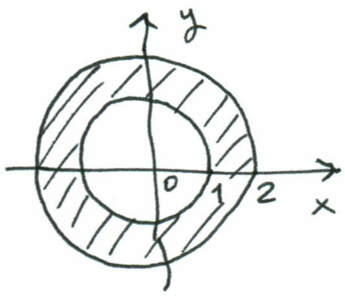
$$= \int_0^3 \int_0^{-\frac{2}{3}x+2} x(6-2x-3y)\,dy\,dx = \cdots = 9$$

5) Find the volume of the region S that lies below the plane $z=x+2$ above the xy-plane and between the cylinders $x^2+y^2=1$ and $x^2+y^2=4$. (Use cylindrical coordinates)

<u>Solution</u> $V=\iiint_S dV$

starting by getting the range for z in terms of cylindrical coordinates; $0 \le z \le x+2 \Rightarrow 0 \le z \le r\cos\theta + 2$

$$\therefore V = \int_0^{2\pi} \int_1^2 \int_0^{r\cos\theta+2} r\,dz\,dr\,d\theta = \cdots \Rightarrow$$



6) Convert $\int_{-1}^{1} \int_0^{\sqrt{1-y^2}} \int_{x^2+y^2}^{\sqrt{x^2+y^2}} xyz\,dz\,dx\,dy$

into an integral in cylindrical coordinates.

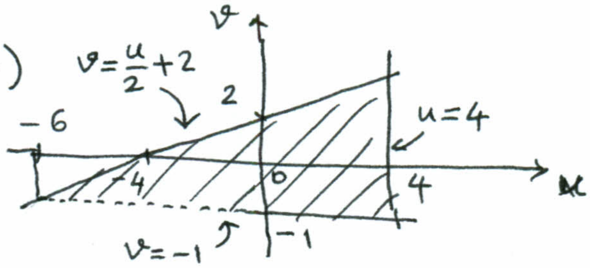
<u>Solution</u> $\int_{-\pi/2}^{\pi/2} \int_0^1 \int_{r^2}^{r} r(r\cos\theta)(r\sin\theta)z\,dz\,dr\,d\theta$

7) Determine the new region that we get by applying the given transformation to the region R;

a) R is the ellipse $36x^2+y^2=36$ and the transformation $u=2x$, $v=\frac{1}{3}y$.

b) R is the region bounded by $y=4-x$, $y=x+1$ and $3y=x-4$ and the transformation is $\left\{x=\frac{u+v}{2},\ y=\frac{u-v}{2}\right\}$.

<u>Solutions</u> a) $u^2+v^2=4$ b)



8) Evaluate $\iint_R (x+y)\,dA$ where R is the trapezoidal region with vertices $(0,0),(5,0),\left(\frac{5}{2},\frac{5}{2}\right),\left(\frac{5}{2},-\frac{5}{2}\right)$ using the transformation $x=2u+3v$, $y=2u-3v$.

<u>Solution</u>

$$=\int_0^{5/6}\int_0^{5/4}(2u+3v)+(2u-3v).\,|-12|\,du\,dv=\cdots=\frac{125}{4}$$

9) Evaluate $\iint_R (x^2-xy+y^2)\,dA$ where R is the ellipse given by $x^2-xy+y^2=2$ and using the transformation $x=\sqrt{2}u-\sqrt{\frac{2}{3}}v$, $y=\sqrt{2}u+\sqrt{\frac{2}{3}}v$. (<u>Hint</u>. $x^2-xy+y^2=2 \Rightarrow \frac{(x+y)^2}{4}+\frac{3(y-x)^2}{4}=2$)

<u>Solution</u> $x^2-xy+y^2=2 \Rightarrow$

$\Rightarrow \cdots \Rightarrow 2u^2+2v^2=2$

or $u^2+v^2=1$. (Because $u^2=\frac{(x+y)^2}{8}$, $v^2=\frac{3(y-x)^2}{8}$)

and $\left|\frac{\partial(x,y)}{\partial(u,v)}\right|=\frac{4}{\sqrt{3}}$

$$\therefore \iint_R (x^2-xy+y^2)\,dA=\iint_{R^*}2(u^2+v^2).\frac{4}{\sqrt{3}}\,du\,dv=\int_0^{2\pi}\int_0^{1}r^2\,r\,dr\,d\theta\left(\frac{8}{\sqrt{3}}\right)$$

$$=\cdots=\boxed{\frac{4\pi}{\sqrt{3}}}$$

10) Find the line integral $\oint_C \vec{F}.d\vec{r}$, where

$$F(x,y) = (\sin y - y\sin x)\vec{i} + (\cos x + x\cos y)\vec{j}$$

and C is the path parametrized as $\vec{r}(t) = e^t \sin t\,\vec{i} + e^t \cos t\,\vec{j}$, $0 \le t \le 4\pi$.

Solution

$$\oint_C \vec{F}.d\vec{r} = \int_C \overset{M}{(\sin y - y\sin x)}dx + \overset{N}{(\cos x + x\cos y)}dy$$

$\frac{\partial N}{\partial x} = -\sin x + \cos y$, $\frac{\partial M}{\partial y} = \cos y - \sin x$. $\therefore$ $\vec{F}$ is a conservative force field and C is a closed set.

$$\therefore \quad \oint_C \vec{F}.d\vec{r} = \boxed{0}$$

11) Evaluate $\iint_R \frac{x+2y}{(2x-y)^2}\,dx\,dy$ over the parallelogram R enclosed by the lines $x+2y=3$, $x+2y=5$, $2x-y=-6$ and $2x-y=-3$. (Hint. Use $u=x+2y$, $v=2x-y$)

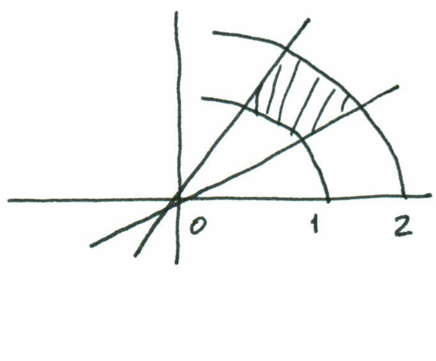
12) Compute $\iiint_S 6xy\,dV$ where S is the region $0 \le z \le 1+x+y$, and x, y are bounded by $y=\sqrt{x}$, $y=1$, $x=0$.

Solution

$$\int_0^1 \int_0^{y^2} \int_0^{1+x+y} 6xy\,dz\,dx\,dy = \cdots = \frac{1}{2} + \frac{1}{4} + \frac{3}{7}$$

13) Let $R = \{(x,y) : 1 \le x^2+y^2 \le 4,\ \frac{1}{\sqrt{3}} \le \frac{y}{x} \le \sqrt{3}\}$ be the region in the first quadrant.

Evaluate $\iint_R \sqrt{x^2+y^2}\,dA$



Solution

$$= \int_{\pi/6}^{\pi/3} \int_1^2 (r)(r\,dr)\,d\theta$$

14) For each of the following vector fields, check whether it is conservative or not. If it is, find a potential function.

a) $\vec{F}(x,y) = 2x^2 \cos(xy)\,\vec{i} + x^3 \cos(xy)\,\vec{j}$

b) $\vec{F}(x,y) = (y^2 + yze^x)\,\vec{i} + (2xy + ze^x)\,\vec{j} + ye^x\,\vec{k}$

Solutions a) It is not conservative

b) $\vec{F}(x,y) = P\vec{i} + Q\vec{j} + R\vec{k}$ and $\begin{cases} P_y = 2y + ze^x = Q_x \\ P_z = ye^x = R_x \\ Q_z = e^x = R_y \end{cases}$

$\therefore$ it is a conservative force field. $\exists\, \varphi(x,y,z)$ such that,

$\therefore$ $\dfrac{\partial \varphi}{\partial x} = P = y^2 + yze^x \Rightarrow \varphi(x,y,z) = \int (y^2 + yze^x)\,dx = y^2 x + yze^x + c(y) + c(z)$

$\dfrac{\partial \varphi}{\partial y} = Q = 2xy + ze^x$ | $\dfrac{\partial \varphi}{\partial y} = 2xy + ze^x + c'(y) \Rightarrow c'(y) = 0 \rightarrow c(y) = c_1$

$\dfrac{\partial \varphi}{\partial z} = R = ye^x$ | $\dfrac{\partial \varphi}{\partial z} = ye^x + c'(z) \Rightarrow c'(z) = 0 \rightarrow c(z) = c_2$

we can choose $c_1 = c_2 = 0$.

$\therefore$ $\boxed{\varphi(x,y,z) = xy^2 + yze^x}$ is the potential function.

15) Evaluate $\displaystyle\int_C (y^2 + yze^x)\,dx + (2xy + ze^x)\,dy + ye^x\,dz$

where C is the curve starting $A = (0,1,2)$ to $B = (1,1,0)$.

Solution We know that it is conservative from 14/b.

$\therefore$ $= \varphi(x,y,z)\Big|_A^B = (xy^2 + yze^x)\Big|_{A(0,1,2)}^{B(1,1,0)}$

$= 1 - 2 = \boxed{-1}$

16) Show that $\vec{F}(x,y) = \left(2x - \frac{1}{x} + y\right)\vec{i} + \left(\frac{1}{y} + x\right)\vec{j}$ is conservative. Find a potential function $\varphi(x,y)$ and evaluate

$$\int_C \left(2x - \frac{1}{x} + y\right)dx + \left(\frac{1}{y} + x\right)dy \quad \text{where } C = \begin{cases} x = t \\ y = t^2 \end{cases},\ 1 \le t \le e$$

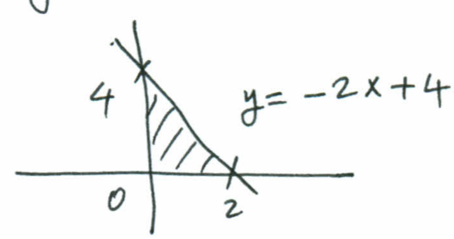
Solution $\varphi(x,y) = x^2 + \ln\left(\frac{y}{x}\right) + xy$ and $\displaystyle\int_C \vec{F}\cdot d\vec{r} = \varphi(x,y)\Big|_{(1,1)}^{(e,e^2)} = \boxed{e^3 + e^2 - 1}$

17) Evaluate $\displaystyle\int_0^1 \int_{\sqrt[3]{y}}^1 \frac{dx\,dy}{1+x^4}$

(Hint. Change the order of integration)

18) Express the volume of a solid bounded by $x=0$, $y=0$, $2x+y+z=4$ and $6x+3y-2z=12$ as a double integral. Do not evaluate the integral.

Solution



xy-projection :
$$4 - 2x - y = \frac{6x + 3y - 12}{2}$$

$$\Rightarrow \cdots \Rightarrow \quad x + \frac{1}{2}y = 2 \Rightarrow y = -2x + 4$$

$$V = \int_0^2 \int_0^{-2x+4} \int_{3x + \frac{3}{2}y - 6}^{4-2x-y} dz\,dy\,dx$$

19) Evaluate $\displaystyle\iint_R 2\frac{y^3}{x^3}\,dA$, where R is the region bounded by $xy=1$, $xy=4$, $\frac{y}{x}=1$, $\frac{y}{x}=4$ in the first quadrant.

Solution $u = xy$, $v = \frac{y}{x}$ $\Rightarrow$ $\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{1}{2v}$

$$= \int_1^4 \int_1^4 \cancel{2} v^3 \cdot \frac{1}{\cancel{2}v}\,du\,dv = \int_1^4 \int_1^4 v^2\,du\,dv = \boxed{4^3 - 1}$$