

Solved Problems

- 1) Write the volume of the region inside the paraboloid $z = x^2 + y^2$ and inside the sphere $x^2 + y^2 + z^2 = 20$. (Use cylindrical coordinates)

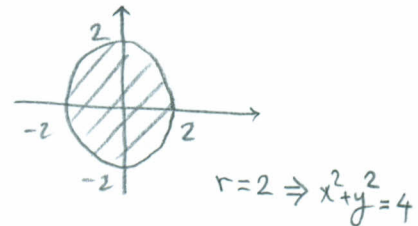
Cylindrical Coordinates

$$\left. \begin{aligned} x &= r \cos \theta \\ y &= r \sin \theta \\ z &= z \\ dV &= r dz dr d\theta \end{aligned} \right\}$$

$$z = x^2 + y^2 \Rightarrow z = r^2$$

$$x^2 + y^2 + z^2 = 20 \Rightarrow r^2 + r^4 = 20 \Rightarrow (r^2 + 5)(r^2 - 4) = 0$$

$$\Rightarrow r = 2$$



$$V = \iiint dV = \int_0^{2\pi} \int_0^2 \int_{r^2}^{\sqrt{20-r^2}} r dz dr d\theta$$

- 2) Evaluate $\int_0^1 \int_{\sqrt{y}}^1 \sqrt{1+x^3} dx dy$

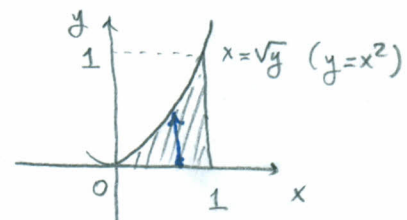
We have to change the order of integration; $R = \begin{cases} x = \sqrt{y}, x = 1 \\ y = 0, y = 1 \end{cases}$

$$= \int_0^1 \int_0^{x^2} \sqrt{1+x^3} dy dx$$

$$= \int_0^1 \sqrt{1+x^3} \cdot x^2 dx$$

$$\left(\begin{aligned} u &= 1+x^3, du = 3x^2 dx \\ x=0 &\rightarrow u=1 \\ x=1 &\rightarrow u=2 \end{aligned} \right)$$

$$= \int_1^2 \sqrt{u} \cdot \frac{du}{3} = \frac{2}{9} (2^{3/2} - 1)$$

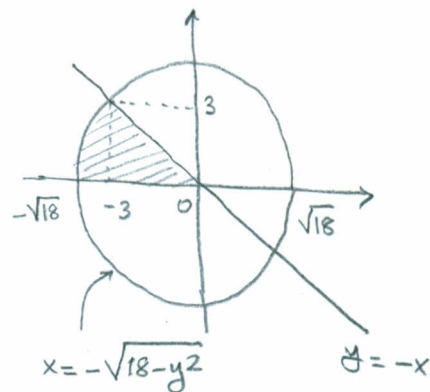


Evaluate $\int_0^3 \int_{-\sqrt{18-y^2}}^0 \cos(x^2+y^2) dx dy$ using polar coordinates.

$$\left. \begin{aligned} x &= -\sqrt{18-y^2} \rightarrow x^2+y^2=18 \\ x &= -y \\ y &= 0, y=3 \end{aligned} \right\}$$

$$\int_{\frac{\pi}{4}}^{\pi} \int_0^{\sqrt{18}} \cos(r^2) \cdot r dr d\theta$$

$$= \int_{\frac{\pi}{4}}^{\pi} \left(\frac{1}{2} \sin(r^2) \Big|_0^{\sqrt{18}} \right) d\theta = \boxed{\frac{\pi}{8} \sin 18}$$



4) Evaluate the line integral

$$\int_C \left(2xy + \frac{1}{x}\right) dx + \left(x^2 - \frac{1}{y}\right) dy \quad \text{from } (-1, -1) \text{ to } (1, 2),$$

where $C : x^3 - xy^2 + y + 1 = 0$.

$\frac{\partial N}{\partial x} = 2y$, $\frac{\partial M}{\partial y} = 2x$. Therefore \vec{F} is conservative and so the line

integral is path independent. Thus, \exists a potential function $\varphi(x, y)$ s.t.

$$\frac{\partial \varphi}{\partial x} = 2xy + \frac{1}{x} \Rightarrow \varphi(x, y) = \int \left(2xy + \frac{1}{x}\right) dx = x^2 y + \ln|x| + c(y)$$

$$\frac{\partial \varphi}{\partial y} = x^2 - \frac{1}{y} \quad \Bigg| \quad \frac{\partial \varphi}{\partial y} = x^2 + c'(y) \Rightarrow c'(y) = -\frac{1}{y} \Rightarrow c(y) = -\ln|y|.$$

Therefore, $\varphi(x, y) = x^2 y + \ln|x| - \ln|y|$ or

$\varphi(x, y) = x^2 y + \ln\left|\frac{x}{y}\right|$ is the potential function.

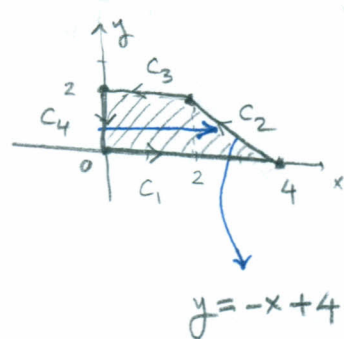
$$\int_C \left(2xy + \frac{1}{x}\right) dx + \left(x^2 - \frac{1}{y}\right) dy = x^2 y + \ln\left|\frac{x}{y}\right| \Bigg|_{(-1, -1)}^{(1, 2)} = \boxed{3 + \ln 2}$$

5) Find the line integral of $\vec{F}(x, y) = (ye^x, xe^{-y})$ over the trapezoid whose vertices are $(0, 0), (0, 2), (2, 2), (4, 0)$ in counter clockwise direction.

$$\oint_C ye^x dx + xe^{-y} dy = ?$$

Using Green's Theorem;

$$\frac{\partial N}{\partial x} = e^{-y}, \quad \frac{\partial M}{\partial y} = e^x$$



$$= \iint_R (e^{-y} - e^x) dA$$

$$= \int_0^2 \left(\int_0^{4-y} (e^{-y} - e^x) dx \right) dy = \int_0^2 \left[\int_0^{4-y} e^{-y} dx - \int_0^{4-y} e^x dx \right] dy$$

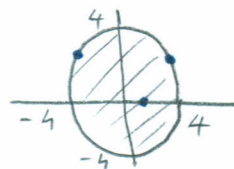
$$= \int_0^2 \left[e^{-y}(4-y) - e^{4-y} + 1 \right] dy = 4 \int_0^2 e^{-y} dy - \int_0^2 ye^{-y} dy - \int_0^2 e^{4-y} dy + \int_0^2 dy$$

$$= 4\left(1 - \frac{1}{e^2}\right) - \left(1 - \frac{3}{e^2}\right) + (e^2 - e^4) + 2 = 5 - 4e^{-2} + 3e^{-2} + e^2 - e^4 = \boxed{5 - e^{-2} + e^2 - e^4}$$

6) Find the extreme values of the function

$$f(x,y) = 2x^2 + 3y^2 - 4x - 5 \quad \text{on the disc } x^2 + y^2 \leq 16.$$

$$\left. \begin{aligned} f_x &= 4x - 4 = 0 \\ f_y &= 6y = 0 \end{aligned} \right\} (1,0) \text{ is a critical point}$$



For the boundary; $x^2 + y^2 = 16 \Rightarrow y^2 = 16 - x^2$

Substituting to the equation we obtain;

$$f(x) = 2x^2 + 3(16 - x^2) - 4x - 5 = -x^2 - 4x + 43.$$

$$f'(x) = -2x - 4 = 0 \Rightarrow x = -2. \quad \therefore (-2, \pm\sqrt{12}) \text{ is also a critical point.}$$

$$y^2 = 16 - 4 = 12.$$

Now, $f(1,0) = 2 - 4 - 5 = -7$, $f(-2, -\sqrt{12}) = 8 + 36 + 8 - 5 = 47$
 $f(-2, \sqrt{12}) = 8 + 36 + 8 - 5 = 47$

$\therefore 47$ is the absolute max. value and -7 is the abs. min. value.

7) Use the method of Lagrange multipliers to find the maximum and minimum value of the function

$$f(x,y,z) = x + 3y - z \quad \text{subject to}$$

the constraint $x^2 + 4y^2 + z^2 = 17.$

Define $F(x,y,z,\lambda) = x + 3y - z - \lambda(x^2 + 4y^2 + z^2 - 17)$. Then,

$$\left. \begin{aligned} F_x &= 1 - 2\lambda x = 0 \\ F_y &= 3 - 8\lambda y = 0 \\ F_z &= -1 - 2\lambda z = 0 \end{aligned} \right\} \lambda = \frac{1}{2x} = \frac{3}{8y} = \frac{-1}{2z}$$

$$\Rightarrow 6x = 8y = -6z \quad \text{or} \quad \begin{aligned} z &= -x \\ y &= \frac{3x}{4} \end{aligned}$$

$$F_\lambda = -(x^2 + 4y^2 + z^2 - 17) = 0$$

Substituting to the constraint, we get

$$x^2 + 4\left(\frac{3x}{4}\right)^2 + (-x)^2 = 17 \Rightarrow \frac{17}{4}x^2 = 17 \Rightarrow x^2 = 4$$

$(\pm 2, \pm \frac{3}{2}, \mp 2)$ is the critical point.

$$f(x,y,z) = x + 3y - z \quad \text{has } \underline{\text{max. value}} \quad 2 + 3\left(\frac{3}{2}\right) - 2 = \boxed{\frac{17}{2}}$$

$$\text{and has } \underline{\text{min. value}} \quad -2 - 3\left(\frac{3}{2}\right) - 2 = \boxed{-\frac{17}{2}}$$

8) Evaluate $\int_C xy dx + z dy + (x-z+1) dz$

if C is the line segment from $P_1(1, -1, 2)$ to $P_2(-1, 2, 2)$.

Since C is not closed and \vec{F} is not conservative, we'll apply the direct method.

Parametric equation of the line segment: $\vec{P} = P_2 + \vec{P_1P_2}t$

$$\Rightarrow (x+1, y-2, z-2) = (-2, 3, 0)t$$

$$\Rightarrow C = \begin{cases} x+1 = -2t \\ y-2 = 3t \\ z-2 = 0 \end{cases}, \quad z-2=0 \quad \text{or} \quad C = \begin{cases} x = -2t-1 \\ y = 3t+2 \\ z = 2 \end{cases}$$

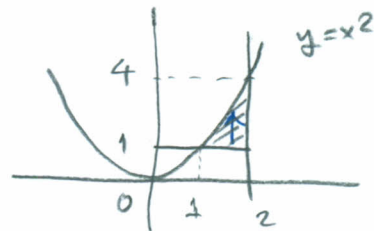
from $t = -1$ to $t = 0$.

$$\int_C xy dx + z dy + (x-z+1) dz$$

$$= \int_{-1}^0 (-2t-1)(3t+2)(-2dt) + 2(3dt) = \int_{-1}^0 (12t^2 + 14t + 10) dt = \dots = \boxed{7}$$

9) Evaluate $\int_1^4 \int_{\sqrt{y}}^2 \sin\left(\frac{x^3}{3} - x\right) dx dy$

$$R = \begin{cases} x = \sqrt{y}, & x = 2 \\ y = 1, & y = 4 \end{cases}$$



Changing the order of integration,

$$= \int_1^2 \int_1^{x^2} \sin\left(\frac{x^3}{3} - x\right) dy dx = \int_1^2 \sin\left(\frac{x^3}{3} - x\right) \cdot (x^2 - 1) dx$$

$$= \int_{-2/3}^{2/3} \sin u du = -\left(\cos u \Big|_{-2/3}^{2/3}\right)$$

$$u = \frac{x^3}{3} - x \quad x=1 \rightarrow u = -\frac{2}{3}$$

$$du = (x^2 - 1) dx \quad x=2 \rightarrow u = \frac{2}{3}$$

$$= -\left[\cos\left(\frac{2}{3}\right) - \underbrace{\cos\left(-\frac{2}{3}\right)}_{\cos\left(\frac{2}{3}\right)}\right] = \boxed{0}$$

10) Evaluate $\int_1^2 \int_0^1 \int_y^1 z \sin(\pi x^2) dx dy dz$.

Since we have the integral $\int_y^1 z \sin(\pi x^2) dx$ at the beginning, and we are not able to evaluate it, the best way is to change the order of integration to the form $dy dx dz$.

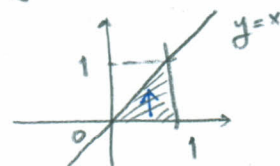
$$\int_1^2 \left(\int_0^1 \int_0^x z \sin(\pi x^2) dy dx \right) dz$$

$$= \int_1^2 \left(\int_0^1 x z \sin(\pi x^2) dx \right) dz$$

$$= \int_1^2 z \left(\frac{-\cos(\pi x^2)}{2\pi} \Big|_0^1 \right) dz = -\frac{1}{2\pi} (\cos \pi - \cos 0) \cdot \left(\frac{z^2}{2} \Big|_1^2 \right)$$

$$= -\frac{1}{2\pi} (-1 - 1) \cdot \frac{3}{2} = \frac{1}{\pi} \cdot \frac{3}{2} = \boxed{\frac{3}{2\pi}}$$

$$R = \begin{cases} x=y, x=1 \\ y=0, y=1 \end{cases}$$



11) Use the transformation $x = \frac{u+v}{2}$, $y = \frac{v-u}{2}$ to evaluate the integral

$$\iint_R (x-y)(x+y) dx dy$$

where R is the region bounded by $y=x+2$, $y=x$, $y=2-x$, $y=4-x$.

12) Find and classify all the critical points of

$$f(x,y) = \frac{1}{3}x^3 - x + xy^2$$

It is straightforward. Try to do it by yourself. Four critical points are $(0,1)$, $(0,-1)$, $(1,0)$, $(-1,0)$.

13) Express the triple integral in cartesian coordinates in the order $dz dy dx$. (Do not evaluate the integral).

$$\int_0^{\pi/2} \int_1^{\sqrt{3}} \int_1^{\sqrt{4-r^2}} z^2 r^3 \sin \theta \cos \theta dz dr d\theta$$

$$\int_0^1 \int_{\sqrt{1-x^2}}^{\sqrt{3-x^2}} \int_1^{\sqrt{4-x^2-y^2}} xyz^2 dz dy dx + \int_1^{\sqrt{3}} \int_0^{\sqrt{3-x^2}} \int_1^{\sqrt{4-x^2-y^2}} xyz^2 dz dy dx$$

14) Let R be the region in xyz -space defined by

$$R = \begin{cases} 1 \leq x \leq 2 \\ 0 \leq xy \leq 2, 0 \leq z \leq 1 \end{cases}$$

Evaluate $\iiint_R (x^2 y + 3xyz) dx dy dz$

by applying the transformation $u=x, v=xy, w=3z$.

Region mapped into $1 \leq u \leq 2, 0 \leq v \leq 2, 0 \leq w \leq 3$.

Jacobian is: $J = \frac{1}{3u}$ Therefore, $= \int \int \int_{R^*} (uv + vw) \frac{1}{3u} dv dw du$

$$= \int_1^2 \int_0^3 \int_0^2 (uv + vw) \frac{1}{3u} dv dw du = \dots = \boxed{2 + 3 \ln 2}$$

15) Write the integral

$$\int_0^2 \int_0^2 f(\sqrt{x^2+y^2}) dx dy$$

in polar coordinates.

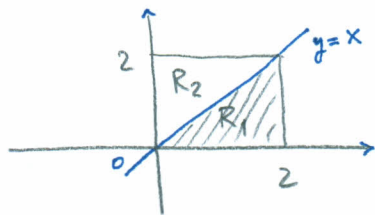
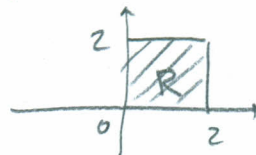
$$\int_0^{\pi/4} \int_0^{\frac{2}{\cos \theta}} f(r) r dr d\theta$$

from R_1

$$+ \int_{\pi/4}^{\pi/2} \int_0^{\frac{2}{\sin \theta}} f(r) r dr d\theta$$

from R_2

$$R = \begin{cases} x=0, x=2 \\ y=0, y=2 \end{cases}$$



$$x=2 \Rightarrow r \cos \theta = 2 \\ \Rightarrow r = \frac{2}{\cos \theta}$$