MCS 152

Exercises (for MT1)

1) Test the following series for convergence. Write the name of the test that you have used.

a) $\sum_{n=1}^{\infty} \frac{7-2n}{5n+1}$ b) $\sum_{n=1}^{\infty} \frac{5^n (n!)^2}{(2n)!}$ c) $\sum_{n=1}^{\infty} \frac{n+3}{(n+1)(n^2+9)}$

2) Find the interval of convergence of $\sum_{n=1}^{\infty} (-1)^n \frac{(x+2)^n}{n 2^n}$

3) Find the sum of the series $\sum_{n=1}^{\infty} \frac{(2+3^n)}{5^{n+3}}$.

4) Given the vectors $\vec{u} = \vec{j} - 3\vec{k}$, $\vec{v} = \vec{i} + 3\vec{j} - \vec{k}$

a) Find $2\vec{u} \cdot (\vec{u} - 2\vec{v})$

b) Find the area of the triangle if two of its sides are $\vec{u}$ and $\vec{v}$

5) Find Maclaurin expansion for each of the following functions

a) $f(x) = \frac{1}{(2-x)^2}$ b) $g(x) = \sin\left(3x + \frac{\pi}{2}\right)$

6) Is the function $f(x,y) = \begin{cases} \frac{5xy}{x^2+2y^2} & : (x,y) \neq (0,0) \\ 0 & : (x,y) = (0,0) \end{cases}$ continuous at the point $(0,0)$?

7) Evaluate the following limit if it exists;

$$\lim_{(x,y)\to(2,3)} \frac{3x^2y - 2xy^2}{9x^2 - 4y^2}$$

8) Let $w = f(x,y,z)$, where $x = 2t^2 + s^5$, $y = e^{st}$, $z = \sqrt{t} \cdot s^3$.

Find $\frac{\partial w}{\partial s}$ and $\frac{\partial w}{\partial t}$ in terms of partial derivatives of f.

9) Find the directional derivative of $f(x,y)=\frac{y^2}{1+x}$ at the point $(1,1)$ in the direction $\vec{i}-3\vec{j}$.

10) If $yz^3 - xyz = \frac{x^2 z}{y}$, where z is a function of x and y, find $\frac{\partial z}{\partial y}$.

11) Find and classify the critical points of $f(x,y)=x^3+\frac{1}{2}y^2-3xy$.

12) Find the interval of convergence of $\sum_{n=1}^{\infty} \frac{(-1)^n \cdot (2x-1)^n}{\sqrt{n+1}}$.

13) let $f(x,y)=x^2-y^2$. Find the critical points of the function $f(x,y)$ in the region $R=\{x^2+y^2 \le 1\}$.

14) Find the extreme values of $f(x,y)=x^2-2y^2$ on the boundary $x^2+y^2=1$ of R using Lagrange Multipliers Method.

15) Show that the triangle with vertices $(-1,2,3)$, $(2,6,3)$ and $(10,0,3)$ is right-angled.

16) Find the sum of the following series;

a) $\sum_{n=3}^{\infty} \frac{5^n}{2^{2n}}$ b) $\sum_{n=1}^{\infty} \frac{5+2^n}{5^{n+2}}$

17) Determine whether the following series converges or diverges by using any appropriate test,

a) $\sum_{n=1}^{\infty} \frac{n^2}{1+n\sqrt{n}}$ b) $\sum_{n=1}^{\infty} \frac{e^n}{n!}$ c) $\sum_{n=1}^{\infty} \frac{1}{n(\ln n)^3}$ (use integral test)

d) $\sum_{n=1}^{\infty} \frac{(-1)^n \cdot n}{n^2+3}$

18) Find the interval of convergence of $\sum_{n=1}^{\infty} \frac{(x+3)^n}{n^2+1}$

19) Find the sum of the series $\sum_{n=1}^{\infty} \frac{n}{5^n}$. (Hint. You may use $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, $|x| < 1$)

20) a) $\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{2x^4+3y^4}$ b) $\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{2x^4+y^4}$

21) Given $f(x,y) = \ln(x^2+y^3)$. Find f_x and f_y at $(-1,1)$.

22) Find and classify all critical points of $f(x,y) = y^3 - xy + x^2$.

23) Find the Maclaurin series of $f(x) = \frac{x^3}{1-5x^2}$. Find also the interval of convergence.

24) Find a vector which is perpendicular to both $\vec{u} = \vec{i} - 2\vec{j} + 3\vec{k}$ and $\vec{v} = 2\vec{i} - 2\vec{j} + \vec{k}$

25) Let $f(x,y,z) = x^2y + y^2z - z^2x$. Find $\vec{\nabla} f(1,-1,1)$.

26) Find and classify all critical points of

$$f(x,y) = x^3 + y^3 - 3xy$$

27) Discuss the convergence for the following series;

a) $\sum_{n=5}^{\infty} \frac{\sqrt{n}}{n^2+n+1}$ b) $\sum_{n=3}^{\infty} \frac{1+n^{4/3}}{2+n^{5/3}}$

28) Find the interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(x-7)^n}{n^5}$$

29) Find a series expansion for $f(x) = \dfrac{x}{(1+x)^2}$

30) Find the interval of convergence of

$$\sum_{n=0}^{\infty} \frac{(n^2+1)(x-2)^n}{(n+3)!}$$

31) Find the extreme values of $f(x,y,z) = 2x - y + z$ over the sphere $x^2 + y^2 + z^2 = 4$.

32) Let $f(x,y) = \sin(x+y) + y \ln(2x+1)$, $x = s - t^2$, $y = e^{st} + 1$.
Find $\dfrac{\partial f}{\partial s}$ and $\dfrac{\partial f}{\partial t}$ in terms of s and t.

33) Let $\vec{u} = \vec{i} - 2\vec{j}$ and $\vec{v} = \vec{i} + \vec{j}$. If $D_{\vec{u}} f(1,2) = 2\sqrt{3}$ and $D_{\vec{v}} f(1,2) = 3\sqrt{2}$, find $f_x(1,2)$ and $f_y(1,2)$.

34) Determine whether $f(x,y)$ is continuous at the point $(0,0)$, where

$$f(x,y) = \begin{cases} \dfrac{x^2 y}{x^3 + xy^2} & : (x,y) \neq (0,0) \\ 0 & : (x,y) = (0,0) \end{cases}$$

35) Find and classify all the critical points of

$$f(x,y) = \frac{2}{3} x^3 + x^2 y + 2y^2 - 5y + 4$$

36) Evaluate the following improper integrals, if exists,

a) $\displaystyle\int_5^{\infty} \frac{1}{x \ln x}\, dx$ b) $\displaystyle\int_0^{\infty} \frac{x+1}{e^x - x}\, dx$

37) Show that for $x \geq 1$, $e^{-x^2} \leq e^{-x}$.
Decide whether the integral $\displaystyle\int_0^{\infty} e^{-x^2}\, dx$ is convergent?